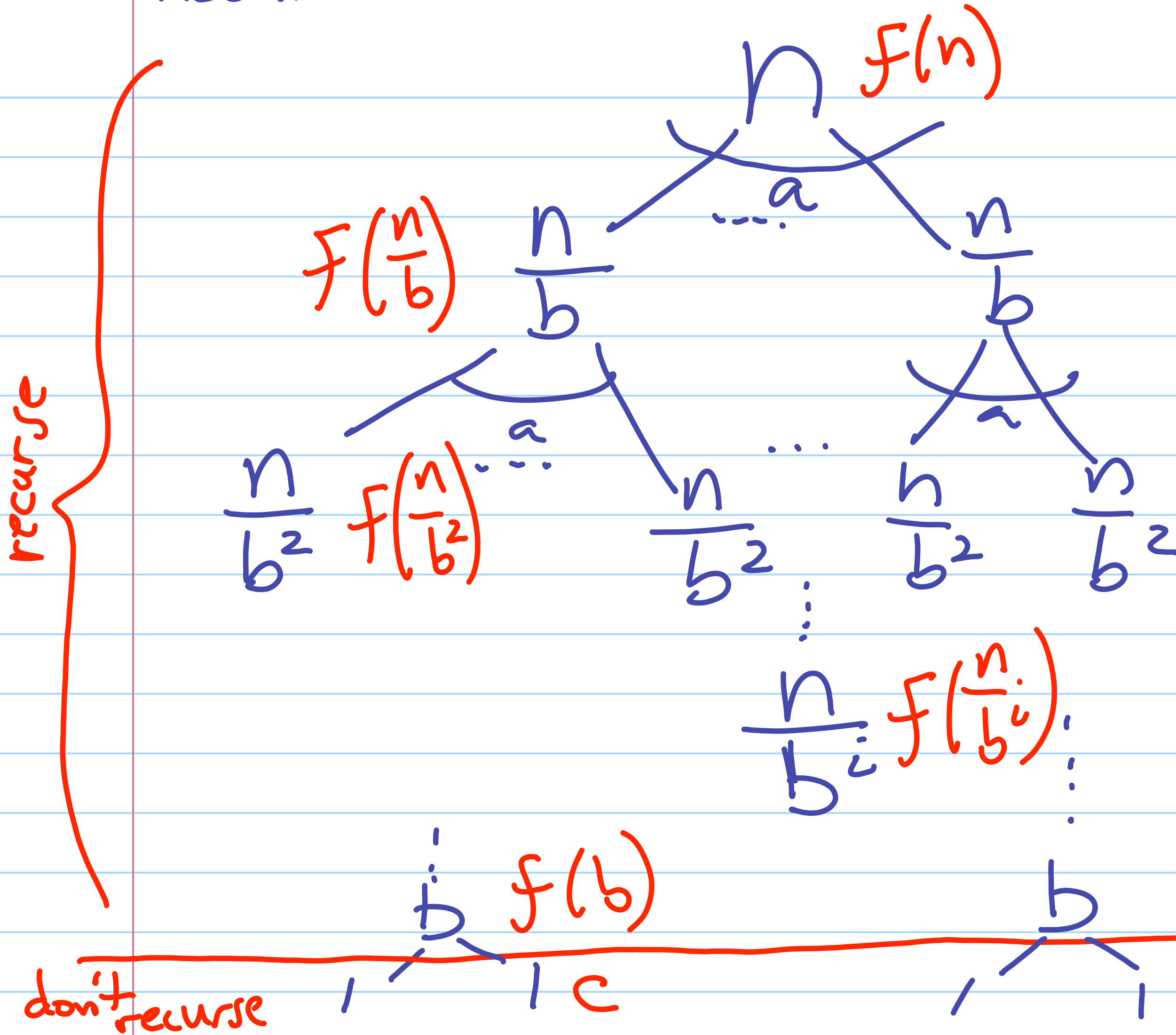
How can you exactly solve a recurrence of the form

$$T(1) = \underline{c}$$

$$T(n) = aT(n/b) + f(n)$$

where n is a power b

Recurrence Tree



level	#	time each
0	1	$\times\, f(n)$
1	a	$\times\, f\left(\frac{n}{b}\right)$
2	a^2	$\times\, f\left(\frac{n}{b^2}\right)$
i	a^i	$\times\, f\left(\frac{n}{b^i}\right)$
$(\log_b n)-1$		$f(b)$
$\log_b n$	$+\ a^{\log_b n}$	$\times\ c$

recurse

level #	# nodes		time per node
0	$a^0 = 1$	$\times$	$f(n)$
1	$a^1 = a$	$\times$	$f(n/b)$
2	a^2	$\times$	$f(n/b^2)$
$\vdots$	$\vdots$	$\vdots$	
i	a^i	$\times$	$f(n/b^i)$
$\vdots$	$\vdots$		$\vdots$
$(\log_b n)-1$	$a^{(\log_b n)-1}$		
$\log_b n$	$a^{\log_b n}$	$\times$	c

$$a^{\log_b n} = \boxed{n^{\log_b a}}$$

verify by taking $\log_b$ of both sides

$$T(n) = \left[\sum_{i=0}^{(\log_b n)-1} a^i \times f(n/b^i) \right] + n^{\log_b a} \cdot c$$

Master Method $F(n) = \Theta\left(n^{\ell}(\log n)^{k}\right)$

$$T(n) = \left[\sum_{i=0}^{(\log_b n)-1} a^i \, \Theta\left(\left(\frac{n}{b^i}\right)^{\ell} \cdot \left(\log \frac{n}{b^i}\right)^{k} \right) \right] + n^{\log_b a} \cdot c$$

$$\log \frac{n}{b^i} = \log n - \log b^i$$

$$= \log n - i \log b$$

$b=2$
$a=2$

$T(1)=1$

$$T(n)=2T(n/2)+\underbrace{cn}_{f(n)}$$

$$T(n)=\left[\sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)\right]+\overset{T(1)}{1}\cdot n^{\log_b a}$$

$$=\left[\sum_{i=0}^{(\log_2 n)-1} \cancel{2^i}\cdot c\cdot \frac{n}{\cancel{2^i}}\right]+n$$

$$=cn\cdot\log_2 n+n$$ } exact for n a power of 2

$T(1) = 1, \quad T(n) = 4T(n/2) + cn$

$a = 4, \; b = 2$
$\log_b a = 2$

$$T(n) = \left[\sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)\right] + 1 \cdot n^{\log_b a}$$

$$= \left[\sum_{i=0}^{(\log_2 n)-1} 4^i \frac{c \cdot n}{2^i}\right] + n^2$$

n

$$= \left(cn \sum_{i=0}^{(\log_2 n)-1} 2^i\right) + n^2 = cn\left(2^{\log_2 n} - 1\right) + n^2$$

$$= cn(n-1) + n^2 = \boxed{(c+1)n^2 - cn}$$

geometric sum $\displaystyle\sum_{i=0}^{m} x^i = \frac{x^{m+1} - 1}{x - 1}$ $\quad x \neq 1$

Does $T(n) = 2T(n/2) + \Theta(n/\log n)$ fit into master method?

No since we would need to set k to -1 & there's a restriction that $k \geq 0$.

$T(n) = T(n-1) + 10, \quad \underline{T(1) = 1}$

recurse

$$\begin{array}{cc} n & 10 \\ n-1 & 10 \\ n-2 & 10 \\ \vdots & \\ 2 & 10 \end{array}$$

how many

terminate

$$1 \qquad +1$$

$T(n) = 10(n-2+1)+1$

$= 10n - 10 + 1$

$\boxed{= 10n - 9}$

Aside

# of numbers $x, x+1, \ldots, y$

$y - x + 1$

$T(n) = T(n-1) + 10, \quad T(1) = 1$

Claim $T(n) = 10n - 9$

Sanity check

	$10n-9$
$T(1) = 1$	$10 - 9 = 1$ ✓
$T(2) = T(1) + 10 = 11$	$20 - 9 = 11$ ✓
$T(3) = T(2) + 10 = 21$	$30 - 9 = 21$ ✓

You can prove correctness using mathematical induction