Analysis

What is height (# of levels in tallest tower)?

X

How many times per level do we expect to move (or peak forward)?
Let \( n \) be \# elements (towers)

Claim: expect \( n \cdot p^i \) elements in level \( i \) list

\[
\begin{align*}
\text{i} = 0 & \quad n \\
\text{i} = 1 & \quad n \cdot p \\
\text{i} \geq 2 & \quad n \cdot p^2 \\
\end{align*}
\]

For \( p = \frac{1}{4} \)

Can prove by induction.
When is \( n \cdot p^i = \frac{1}{p} \)?

Solve for \( i \):

\[ n = \left( \frac{1}{p} \right)^{i+1} \Rightarrow i + 1 = \log_{\frac{1}{p}} n \]

\[ i = \log_{\frac{1}{p}} n - 1 \]

Level when expect \( \frac{1}{p} \) elements to remain

\( p = \frac{1}{4} \)

Expected height:

\[ E[\text{height}] = \log_{\frac{1}{4}} n + \frac{1}{1-p} \]

Tallest tower:

\[ E[\text{height}] = \log_{\frac{1}{4}} n + \frac{4}{3} \]
We’ll finish analysis next time.

Skip List

Overview

<table>
<thead>
<tr>
<th>As a function of $p$</th>
<th>$p = 1/2$</th>
<th>$p = 1/e$</th>
<th>$p = 1/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>space usage</td>
<td>$2n/(1 - p) + 4 \log_{1/p} n$</td>
<td>$4n + 4 \log_2 n$</td>
<td>$\approx 3.2n + 2.8 \log_2 n$</td>
</tr>
<tr>
<td>search cost</td>
<td>$(\log_{1/p} n)/p$</td>
<td>$2 \log_2 n$</td>
<td>$\approx 1.88 \log_2 n$</td>
</tr>
</tbody>
</table>
Ordered Collection Overview

Skip List Analysis

Analysis

(What is expected height (# of levels in tallest tower)?)

x

(How many times per level do we expect to move (or peak) forward)?
Last time argued that expect $n \cdot p^i$ elements in level $i$ list.

Find $i^*$ for which $n \cdot p^i = \frac{1}{p}$

For $p = \frac{1}{4}$, level when you only expect 4 elements to remain.

$n = \left(\frac{1}{p}\right)^{i+1}$

$\log_{1/p} n = i + 1$

$i = \log_{1/p} n - 1$
So at level $\log_{\frac{1}{p}} n - 1$ expect only $\frac{1}{p}$ elements to remain.

Note levels until this point $\log_{\frac{1}{p}} n - 1$

Can prove

$$E[\text{height of tallest tower}] = \log_{\frac{1}{p}} n + \frac{1}{1-p} \left( \log_4 n + \frac{4}{3} \right)$$
Search Cost

Page 615 goes over more formally.

Intuitively,

level \( i \)

\[ \text{look at towers at level } i-1 \]

\( \text{prob } \frac{1}{p} \text{ it reaches level } i \)

\[ \{ \text{Expect to look at } \frac{1}{p} \text{ per level} \} \]
So expect $\log_{1/p} n + \frac{1}{1-p}$ levels

$\Rightarrow E[\text{search time}] \approx \frac{1}{p} \log_{1/p} n = \frac{1}{p} \log_{2} n$

For $p = \frac{1}{4}$, $E[\text{search time}] \approx 2 \log_{2} n$
Space Usage

What is expected height of a tower

\[ E[h] = \sum_{i=0}^{\infty} i \, p^{i-1} (1-p) = \frac{1}{1-p} \]
# next/prev refs (dominates space usage)

\[
\sim N \cdot \frac{1}{1-P} \cdot \frac{2}{\text{expected height}} + 4 \left( \log \frac{n}{1-P} + \frac{1}{1-P} \right)
\]

\[
= \frac{2n}{1-P} + 4 \log \frac{n}{1-P} + \frac{4}{1-P}
\]
# next/prev refs in skip list for \( p = \frac{1}{4} \)

\[ 2^{\frac{2}{3}} n + 2 \log_2 n + \frac{8}{3} \approx 2^{\frac{2}{3}} n \]

# left, right, parent refs in red-black tree

\[ 3n \]

# parent, child, next, prev refs in \( \mathbb{B}^+ \)-tree with \( t = 2 \)

On average \( \approx n/2 \) leaves + \( \approx \frac{3n}{4} \) internal nodes

\[ \Rightarrow \approx 3(\frac{n}{2}) + 5(\frac{3n}{4}) = \frac{21n}{4} = 5\frac{1}{4} n \]
**Skip List overview**

<table>
<thead>
<tr>
<th>As a function of $p$</th>
<th>$p = 1/2$</th>
<th>$p = 1/e$</th>
<th>$p = 1/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>space usage</strong></td>
<td>$2n/(1-p) + 4 \log_{1/p} n$</td>
<td>$4n + 4 \log_2 n$</td>
<td>$\approx 3.2n + 2.8 \log_2 n$</td>
</tr>
<tr>
<td><strong>search cost</strong></td>
<td>$(\log_{1/p} n)/p$</td>
<td>$2 \log_2 n$</td>
<td>$\approx 1.88 \log_2 n$</td>
</tr>
</tbody>
</table>

↑ value of $p$ to minimize search cost