Asymptotic Notation

\[ 6n \log_2 n = O(n^2) \]

Graph:
- \( y = 6n \log_2 n \)
- \( y = n^2 \)

Not much difference

difference gets larger + larger
While $n$ doesn't grow arbitrarily large, it's useful to consider when it does

$$\lim_{n \to \infty} \frac{6n \log_2 n}{n^2} = 0$$

$$+ \lim_{n \to \infty} \frac{n^2}{6n \log_2 n} = \infty$$
In general

\[
\lim_{n \to \infty} \frac{F(n)}{g(n)} = \begin{cases} 
0 & \text{if } f(n) \text{ is asymptotically slower growing than } g(n) \ 
\text{constant} & \text{if } f(n) \text{ and } g(n) \text{ grow at same asymptotic rate} \ 
\infty & \text{if } f(n) \text{ is asymptotically faster growing than } g(n) 
\end{cases}
\]

think of as time complexities
assume \( f(n) \geq 0 \) and \( g(n) \geq 0 \)
**Big-Oh Notation**  asymptotic upper bound

\[ f(n) = O(g(n)) \text{ if there exist constants } C, N_0 \text{ such that } \forall n \geq N_0 \quad f(n) \leq C \cdot g(n) \]
\[ \exists s \left( \frac{3}{2} n^2 + \frac{7n}{2} - 4 \leq O(n^2) \right) \]

This is really a shorthand for set containment (\( \subseteq \)).

\( O(n^2) \) is the set of functions \( \{ f(n) \mid f(n) \text{ is } O(n^2) \} \) such that:

\[ \forall n \geq 6 \quad \frac{3}{2} n^2 + \frac{7n}{2} - 4 \leq 2n^2 \]
\[ T(n) = 2T \left( \frac{n}{2} \right) + O(n) \]

More exs asymp. "\leq"

\[ 6n \log_2 n = O(n^3) \]

Is it meaningful to say

\[ T(n) \geq O(n^2) ? \]

\[ 1 = O(n^2) \text{ all we can conclude is } Tcn) \leq 1 \]
Is it meaningful to say $T(n) \leq O(n^2)$? Yes.

$T(n)$ asymptotically grows no faster than $c \cdot n^2$.

The same as $T(n) = O(n^2)$.

Is $cn^3 = O(n^2)$? No.
When \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \) or constant

then \( f(n) = O(g(n)) \)

When \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \) then

\( f(n) \neq O(g(n)) \)
Big Omega Notation (asymptotic lower bound)

$f(n) = \Omega(g(n))$ if there exists a constant $c > 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq N_0$
\[ f(n) = O(g(n)) \]

Simple mathematical function: \( n, n^2, n \log n, \sqrt{n} \)

Time complexity

Time complexity to find closest pair of points among \( n \) points = \( \sqrt{\frac{n \log_2 n}{2}} \)
\[ f(n) = \lim_{n \to \infty} g(n) \]

equiv \[ f(n) \geq \lim_{n \to \infty} g(n) \]

\[ f(n) \leq \lim_{n \to \infty} g(n) \]

Big-Theta

\[ f(n) = \Theta(g(n)) \text{ is defined as } \]

\[ f(n) = O(g(n)) \text{ and } \Omega(g(n)) \]
\[ \leq \lim_{n \to \infty} \] asymptotic bound (within a constant)

\[ f(n) \leq \lim_{n \to \infty} g(n) \quad f(n) = O(g(n)) \]

\[ f(n) < \lim_{n \to \infty} g(n) \quad f(n) = o(g(n)) \]

\[ f(n) = \lim_{n \to \infty} g(n) \quad f(n) = \Theta(g(n)) \]

\[ f(n) \geq \lim_{n \to \infty} g(n) \quad f(n) = \Omega(g(n)) \]

\[ f(n) > \lim_{n \to \infty} g(n) \quad f(n) = \omega(g(n)) \]
\[ F(n) = O(g(n)) \quad \text{think} \quad F(n) \leq \lim_{n \to \infty} g(n) \]
\[ g(n) = O(h(n)) \quad g(n) \leq \lim_{n \to \infty} h(n) \]
\[ F(n) = O(h(n)) \quad f(n) \leq \lim_{n \to \infty} h(n) \]
\[ g(n) = \Omega(f(n)) \quad g(n) \geq \lim_{n \to \infty} f(n) \]
Overview

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