How can we know when our algorithm is optimal (asymptotically)?

Is there a sorting algorithm with asymptotic time complexity (worst-case or expected case) better than $O(n \log n)$?
We can't prove any limitation (lower bound) without basing it on some model of computation.

Model of Computation

Comparison-based model: you can only learn about relative order of elements through a comparison.
Observation:
For a comparison-based alg.
\[ \text{time complexity} \geq \# \text{ of comparisons} \]

Can I prove a statement of form: Any comparison-based alg to sort \( n \) elements makes \( \geq F(n) \) comparisons?
Adversary Lower Bound

View as a game with 2 players

"Picks" a # between 1 & 100

Adversary (Devil D)

Round

Is your # <= x

Yes or no (can't "lie")

Comparison-based Algorithm A to play "20 question"
Define adversary strategy
Must describe how to reply to whatever question the alg asks in a way that all answers are consistent with some "input" picked

Goal of adv: max # rounds (one question answer/round)
Alg: tries to minimize the # of rounds
What's a good adv. strategy for 20 questions.

Adv can make a list L with all possible #s

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Alg: is # < 9?

- yes means 1-8 "alive"
- no means 9-10 "alive"

Is # < 4?

- yes 1, 2, 3 alive
- no 4, 5, 6, 7, 8 alive
Correct

A\text{ly} cannot be done until $|Z| = 1$.

Why not? Whatever the alg says the answer is, the adv. can report that he had a different answer all along.

This answer is an input that causes the # of comparisons to be # of rounds in the game.
Goal:
For any comparison-based alg to solve problem $P$ there exist an input of size $n$ for which computation time $\geq f(n)$.
Analyze adv. strategy we gave for "20 questions" where adv picks 1, …, n

Initially |L| = N

Question: How many rounds must occur before |L| could be 1. (maybe more rounds are needed)
Initially, $|L_i| = n$

If $L_u$ is # of elements in $|L_i|$ after round $i$ ( $L_0 = n$ )

$L_{i+1} \geq \lceil L_u / 2 \rceil$

# rounds until $|L_i| = 1 \geq \lceil \log_2 n \rceil$
\[ \begin{align*}
\text{Ex:} & \quad Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \\
& \\n& n = 16, 8, 4, 2, 1 \\
& \\
& \log_2 16 = 4 \\
& \\
& n = 17, 9, 5, 3, 2, 1
\end{align*} \]